CONTACT SHEAR

by

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Abstract:

In this paper a problem is introduced between current mathematical solutions in elasticity and physical evidence. This problem is apparent when the contact shear underneath the imposed load is disregarded; assuming the imposed load on the surface is not a point load. A different answer from various solutions in the semi-infinite solid of mass is revealed when starting with a line load solution and extending it to an infinite uniform load. This problem is explained to effect solutions to the boundary value problem for linear and non linear material. The contact shear is shown, at least in the case discussed in this paper, that it cannot be ignored.

Introduction:

Because historically every physicist and engineer ignored the static contact shear and its effect the author declines to elaborate in an introduction. The presented article is a major eye opener to why materials deteriorate on contact.

Mathematical Differences:

Case 1- When starting with a line load acting within an infinite solid (Integrated Kelvin [1] problem I), Fig1.a. the stresses in the solid are:

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$$\sigma_z = \frac{p}{\pi(1-\nu)}\frac{z}{R^2}\left[\frac{3-2\nu}{2} - \frac{x^2}{R^2}\right] \quad \text{.. (1)}$$

$$\sigma_x = \frac{p}{\pi(1-\nu)}\frac{z}{R^2}\left[-\frac{1-2\nu}{2} + \frac{x^2}{R^2}\right] \quad \text{... (2)}$$

$$\sigma_y = \frac{p\nu}{\pi(1-\nu)}\frac{z}{R^2} \quad \text{.. (3)}$$

$$\tau_{xz} = \frac{p}{\pi(1-\nu)}\frac{x}{R^2}\left[\frac{1-2\nu}{2} + \frac{z^2}{R^2}\right] \quad \text{.. (4)}$$

Where ν is Poisson's ratio and $2p$ is the point load.

These stresses are for a plain strain consideration. Now consider Fig.1.b; by integrating Eq 1, 2, 3 and 4 over a for $-b \le a \le b$, $2p = 2p\ da$ and substituting x by $(x - a)$ yields:

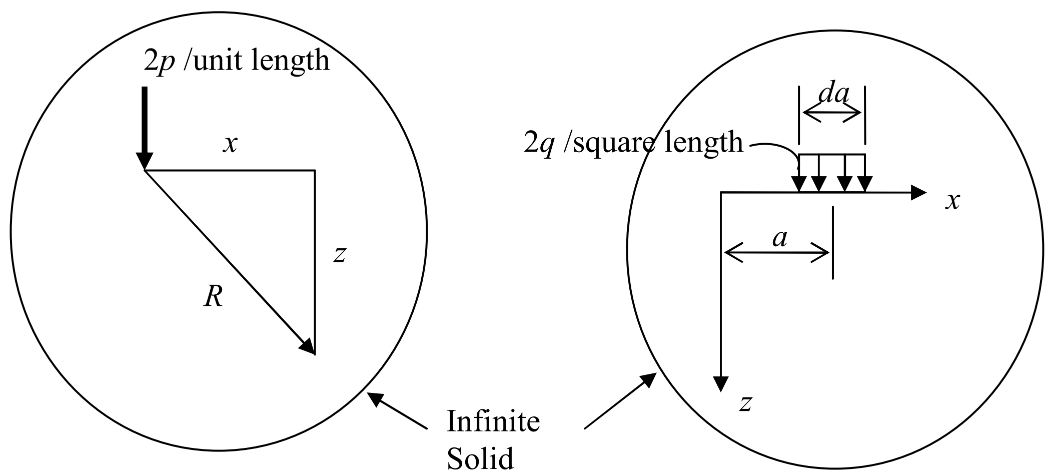

Fig.1.a   Fig.1.b

$$\sigma_z = \frac{q}{\pi}\left(\frac{1}{1-\nu}\right)\left\{(1-\nu)\left[-\tan^{-1}\left(\frac{x-b}{z}\right)+\tan^{-1}\left(\frac{x+b}{z}\right)\right]+\frac{1}{2}\frac{z(x-b)}{(x-b)^2+z^2}+\frac{1}{2}\frac{z(x+b)}{(x+b)^2+z^2}\right\} \quad \text{(5)}$$

$$\sigma_x = \frac{q}{\pi}\left(\frac{1}{1-\nu}\right)\left\{\nu\left[-\tan^{-1}\left(\frac{x-b}{z}\right)+\tan^{-1}\left(\frac{x+b}{z}\right)\right]+\frac{1}{2}\frac{z(x-b)}{(x-b)^2+z^2}-\frac{1}{2}\frac{z(x+b)}{(x+b)^2+z^2}\right\} \quad \text{(6)}$$

$$\sigma_y = \frac{q}{\pi}\left(\frac{\nu}{1-\nu}\right)\left\{-\tan^{-1}\left(\frac{x-b}{z}\right)+\tan^{-1}\left(\frac{x+b}{z}\right)\right\} \quad \ldots\ldots (7)$$

$$\tau_{xz} = \frac{q}{\pi}\left(\frac{1}{1-\nu}\right)\left\{-\left(\frac{1-2\nu}{4}\right)\ln\left[\frac{(x-b)^2+z^2}{(x+b)^2+z^2}\right]+\frac{1}{2}\frac{z^2}{(x-b)^2+z^2}-\frac{1}{2}\frac{z^2}{(x+b)^2+z^2}\right\}$$

$$\ldots\ldots (8)$$

Now let $b \to \infty$ for a stationary point x, yields,

$$\sigma_z = \pm q, \qquad \sigma_x = \pm q\left(\frac{\nu}{1-\nu}\right), \qquad \sigma_y = \pm q\left(\frac{\nu}{1-\nu}\right) \quad \text{and} \quad \tau_{xy} = 0 \quad \ldots\ldots (9)$$

Where the plus sign is compression for $z > 0$, the minus sign is tension for $z < 0$ and at $z = 0$ $\sigma_z = 2q$.

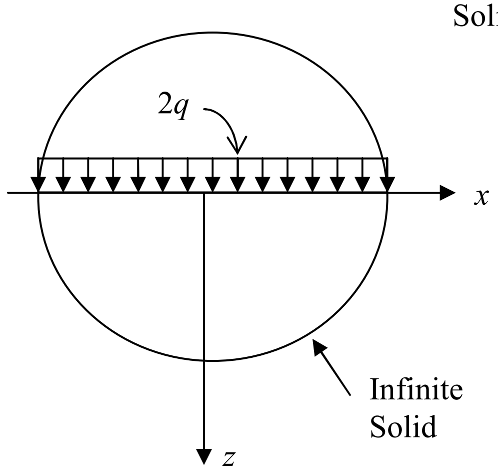

Fig.2.a

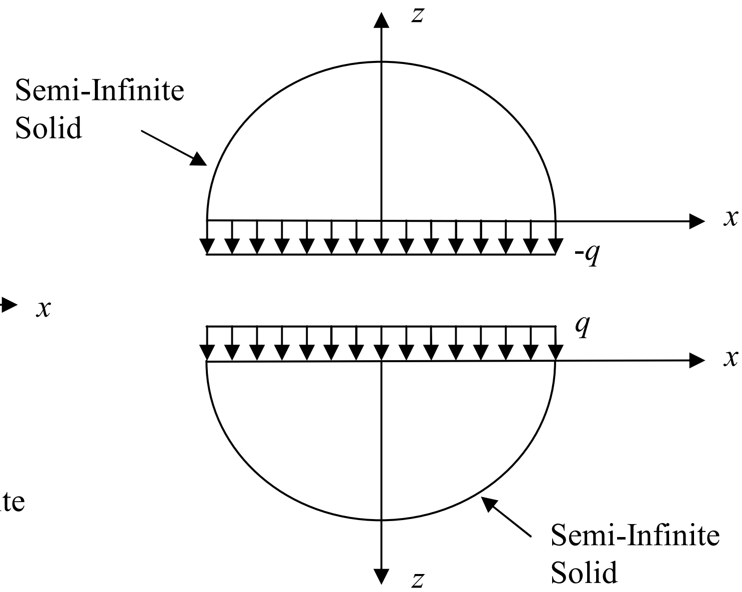

Fig.2.b

Fig 2.a shows the infinite uniform load inside the solid. By taking a free-body diagram at the x-axis the stresses on a semi-infinite mass of solid is obtained; as per Fig.2.b. Thus;

$$\sigma_z = q, \qquad \sigma_x = \sigma_y = K_0 q \quad \text{and} \quad \tau_{xy} = 0 \quad \text{where} \quad K_0 = \frac{\nu}{1-\nu} \quad \ldots\ldots\ldots\ldots\ldots.. (10)$$

Case 2 – When starting with a line load acting on the surface of a semi-infinite mass (Integrated Boussinesque [2] problem) Fig.3.a the stresses in the solid are:

$$\sigma_z = \frac{2p}{\pi}\frac{z^3}{R^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$\sigma_x = \frac{2p}{\pi}\frac{x^2 z}{R^4} \quad \ldots\ldots (12)$$

$$\sigma_y = \nu\frac{2p}{\pi}\frac{z}{R^2} \quad \ldots\ldots (13)$$

$$\tau_{xz} = \frac{2p}{\pi}\frac{xz^2}{R^4} \quad \ldots\ldots (14)$$

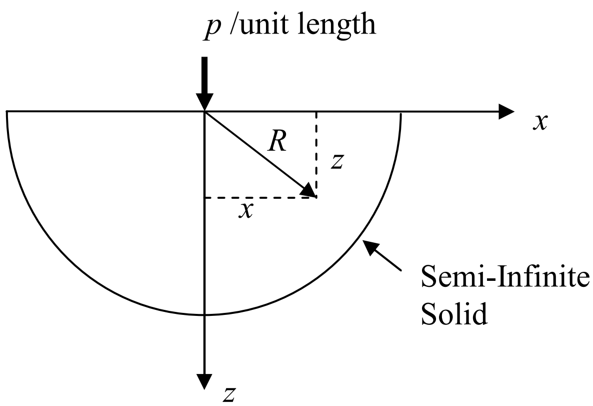

Fig.3.a

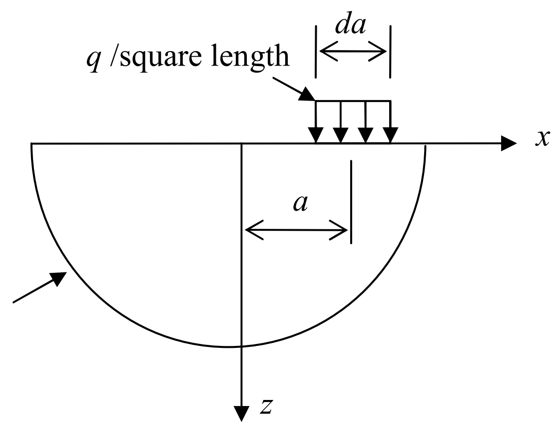

Fig.3.b

These stresses are for plain strain consideration. Consider Fig.3.b; by integrating Eq. 11, 12, 13 and 14 over a for $p = q\,da$ and substituting x by $(x\text{-}a)$ in the interval $-b \le a \le b$ yields:

$$\sigma_z = \frac{2q}{\pi}\left\{\frac{1}{2}\left[-\tan^{-1}\left(\frac{x-b}{z}\right)+\tan^{-1}\left(\frac{x+b}{z}\right)\right]-\frac{1}{2}\frac{z(x-b)}{(x-b)^2+z^2}+\frac{1}{2}\frac{z(x+b)}{(x+b)^2+z^2}\right\}$$

…………………………………….. (15)

$$\sigma_x = \frac{2q}{\pi}\left\{\frac{1}{2}\left[-\tan^{-1}\left(\frac{x-b}{z}\right)+\tan^{-1}\left(\frac{x+b}{z}\right)\right]+\frac{1}{2}\frac{z(x-b)}{(x-b)^2+z^2}-\frac{1}{2}\frac{z(x+b)}{(x+b)^2+z^2}\right\}$$

…………………………………….. (16)

$$\sigma_y = \frac{2q}{\pi}\nu\left[-\tan^{-1}\left(\frac{x-b}{z}\right)+\tan^{-1}\left(\frac{x+b}{z}\right)\right] \quad \text{…………………………………….. (17)}$$

$$\tau_{xz} = \frac{2q}{\pi}\left[\frac{1}{2}\frac{z^2}{(x-b)^2+z^2}-\frac{1}{2}\frac{z^2}{(x+b)^2+z^2}\right] \quad \text{…………………………………….. (18)}$$

Now let $b \to \infty$ for a stationary point x, yields,

$$\sigma_z = q, \qquad \sigma_x = q, \qquad \sigma_y = 2\nu q \quad \text{and} \quad \tau_{xy} = 0 \quad \text{…………………………..…. (19)}$$

Case 3- When starting with a vertical line load beneath the surface of a semi-infinite mass of solid (Melan's [3] problem I) Fig.4; the stresses in the solid are not shown for simplicity. However when integrating in the same manner as in case 1 and 2 of the above section, the stresses become:

$$\sigma_z = \begin{Bmatrix} 0 & z<d \\ q & z \ge d \end{Bmatrix}, \quad \sigma_x = \begin{Bmatrix} (1-K_0)q & z<d \\ q & z \ge d \end{Bmatrix}, \quad \sigma_y = \begin{Bmatrix} \nu(1-K_0)q & z<d \\ 2\nu q & z \ge d \end{Bmatrix} \text{ and } \tau_{xy} = 0$$

………………………………. (20)

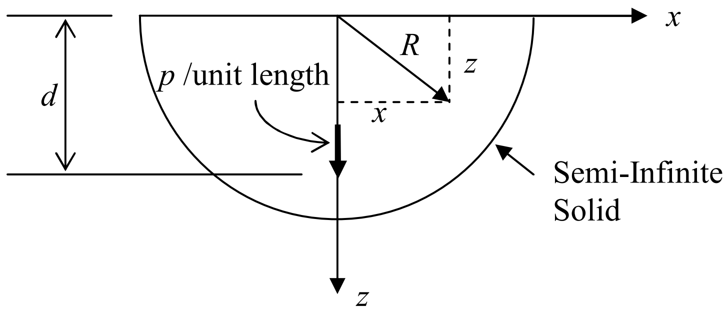

Fig.4

Physical Interpretation:

Definitely case 1 and 2 of the last section gives two different answer to the same problem. The question is Eq. 10 or Eq. 19 is the correct answer? From physical stand point the deflection in the *x* direction is zero everywhere. Therefore, the strain is zero everywhere $\varepsilon_x = 0$ or

$$\varepsilon_x = \beta\sigma_x - \rho\sigma_z = 0 \quad \text{where} \quad \beta = \frac{1-\nu^2}{E} \quad \text{and} \quad \rho = \frac{\nu(1+\nu)}{E} \quad \text{……………….. (21)}$$

Where *E* is the elastic modulus, this loads to:

$$\sigma_x = \sigma_y = \frac{\rho}{\beta}\sigma_z = K_0\sigma_z = K_0 q \qquad (22)$$

Thus Eq. 10 is correct. In case 3 Eq. 20 is not correct since. This can also be understood that the deflection in the *z* direction is the same everywhere and the deflection in the *x* direction is zero. Thus, the strain energy above and below the imposed load is zero everywhere in the solid. Furthermore, σ_x below the imposed load should equal to $K_0\ q$ per Eq. 22 and equal to zero above the load by taking σ_z from Eq. 20. This difference is the subject of this paper and these equations will become complete when considering the contact shear on the surface underneath the load.

Contact Shear

Boussinesque solution for a point load is complete. The problems arise when integrating these equations due to a stress on the surface. Consider Fig. 5

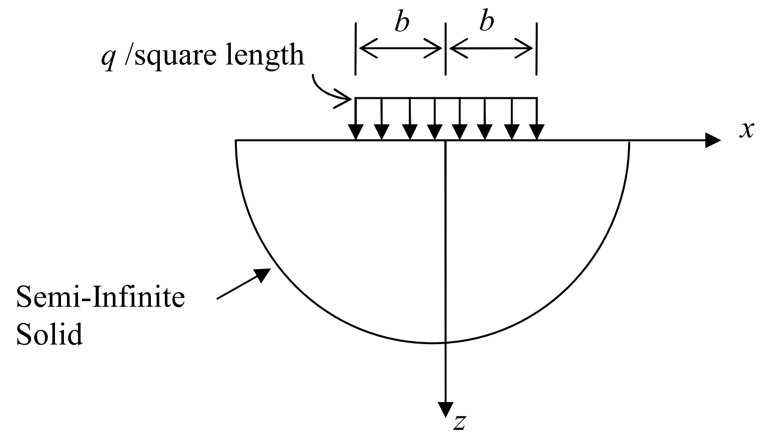

Fig. 5

Note Bousinesque equation gives ε_x at $x = 0$ and $z = 0$ a constant compressive strain value of $q(\beta - \rho)$ which causes extra lateral load and gives a negative deflection of $-q(\beta - \rho)|x|$ at $-b \leq x \leq b$ and zero otherwise and at $\nu = 0.5$ the deflection and the strain is zero everywhere. This physically does not make sense because as $x = b$ is large or approach infinity an infinite lateral deflection occur under the load at b. Also it does not make sense especially when considering the load transmitted by a material. Even if the surface is frictionless as in magnetic levitation or uniform hydrostatic pressure the deflection in the x direction due to uniform load is expected to be zero under the load also the deflection is expected to be zero for a hard material pressing on a softer material or vise versa other wise both materials deflect laterally to infinity at large b. The reason is no mater how the deflection function is defined at the surface underneath the load, if b is large or goes to infinity a large lateral deflection occur that does not make physical sense. Surely the point at the center of the load in Fig.5 and its neighborhood is not going to go any place. This means physically the deflection at $z = 0$ and x in the interval $-\delta \leq x \leq \delta$, where δ is small but not zero, is close to zero. Thus $\varepsilon_x \rightarrow 0$ at least at $z = 0$ and $x = 0$ at b approach infinity. It is important to mention that it does not matter how the load is transferred. For example if the load is transferred though a rigid or a soft type material, still $\varepsilon_x \rightarrow 0$ at $z = 0$ and $x = 0$ for both materials at b approach infinity. Once the load is transferred by a material the deflection will start zero underneath the load and move positively in the direction of x beyond the load as the material continue to press the load. The above criterion is possible. Take for example a material such as rubber with $\nu = 0.5$ and $K_0 = 1.0$; immediately case 1, 2 and 3 above gives $\varepsilon_x = 0$ at the surface everywhere; in this case b did not enter the equation. Why should it be any different for any other material? Therefore, to complete Bousinessque equations, one must consider a contact shear function that cause $\varepsilon_x \rightarrow 0$ at least at $z = 0$ and $x = 0$ for b approaching infinity. Furthermore, this shearing pressure must disappear when $b \rightarrow \infty$ and give an additional stress to Bousinesque equation of magnitude equal to $(K_0 - 1)q$ to σ_x.

If a material touches the surface then deflection compatibles between the semi-infinite solid and the material pressing to transfer the load is required. And therefore the contact shear can be bounded by a load transferred by a rigid surface with no slippage and the contact shear on other materials with no slippage, this contact shear will be derived in this paper. Also the maximum contact shear will be derived in this paper and should be used as companion when integrating any load for b approaching infinity in order to have the correct stresses in the solid.

An immediate consequence to this phenomena is that the contact shear not only must exist for a compressive stress on the surface, but also for a tension type stress, the surface between the load and the semi-infinite mass must be glued and one must consider $\varepsilon_x = 0$ at $x = 0, z = 0$ at b approach infinity.

Analysis:

1-Existence: The task now is to find a shear function $t(x)$ that satisfies the criterion in the previous section. The foregoing analysis is to show that these criterion can be satisfied very easily on the simplest form and can be expanded to any loading function. The analysis will begin with a simple form function $t(x) = A \sin kx$ for the uniform load q as shown in Fig.6 where A and k are constants.

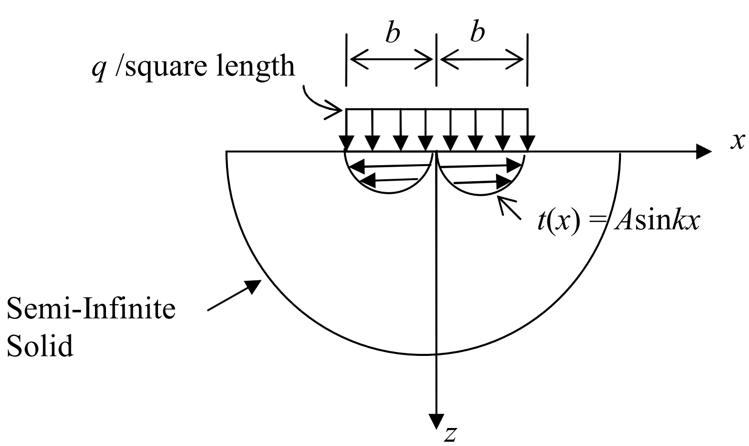

Fig. 6

From Boussinesque equations for a shear point load on the surface the stresses can be found for $t(x)$ as:

$$\sigma_z = \frac{2z^2}{\pi} A \int_{-b}^{b} \frac{(x-a)\sin ka}{\left[(x-a)^2 + z^2\right]^2} da \quad \ldots \quad (23)$$

$$\sigma_x = \frac{2}{\pi} A \int_{-b}^{b} \frac{(x-a)\sin ka}{(x-a)^2 + z^2} da - \sigma_z \quad \ldots \quad (24)$$

$$\tau_{xz} = \frac{2z}{\pi} A \int_{-b}^{b} \frac{(x-a)^2 \sin ka}{\left[(x-a)^2 + z^2\right]^2} da \quad \ldots \quad (25)$$

At $b \to \infty$ yields:

$$\sigma_z = \frac{2z^2}{\pi} A \int_{-\infty}^{\infty} \frac{u \sin k(u-x)}{\left[u^2 + z^2\right]^2} du \qquad \text{with} \qquad u = x - a$$

Or

$$\sigma_z = -\frac{2z^2}{\pi} A \cos kx \int_{-\infty}^{\infty} \frac{u \sin ku}{\left[u^2 + z^2\right]^2} du + \frac{2z^2}{\pi} A \sin kx \int_{-\infty}^{\infty} \frac{\cos ku}{\left[u^2 + z^2\right]^2} du$$

Or

$$\sigma_z = -kzA \cos kx \, e^{-zk} \quad \ldots \quad (26)$$

Similarly:

$$\sigma_x = -\frac{2}{\pi} A\cos kx \int_{-\infty}^{\infty} \frac{u\sin ku}{u^2+z^2}du + \frac{2}{\pi} A\sin kx \int_{-\infty}^{\infty} \frac{\cos ku}{u^2+z^2}du - \sigma_z$$

$$\sigma_x = -2A\cos kx\, e^{-zk} - \sigma_z \quad \text{……………………………………………………………} (27)$$

And

$$\tau_{xz} = A(1-kz)\sin kx\, e^{-kz} \quad \text{……………………………………………………………} (28)$$

Note:

If $k \to 0$ then $\sigma_z = 0$, $\sigma_x = -2A$ and $\tau_{xz} = 0$

If $z = 0$ then $\sigma_z = 0$, $\sigma_x = -2A\cos kx$ and $\tau_{xz} = A\sin kx$

Now if one chooses $k = \frac{\alpha}{b}$ and $A = \frac{q\pi}{4}\left(1-\frac{\rho}{\beta}\right)\frac{1}{SI(\alpha)}$ where $SI(\alpha) = \int_0^{\alpha} \frac{\sin u}{u}du$

then the contact shear becomes:

$$t(x,\alpha) = \frac{\pi}{4}\frac{q}{SI(\alpha)}\left(1-\frac{\rho}{\beta}\right)\sin\left(\frac{\alpha x}{b}\right) \quad \text{…………………………………………………..} (29)$$

And

$$\sigma_z = \frac{z^2}{2}\left(1-\frac{\rho}{\beta}\right)\frac{q}{SI(\alpha)}\int_{-b}^{b}\frac{(x-a)\sin\left(\frac{\alpha a}{b}\right)}{\left[(x-a)^2+z^2\right]^2}da$$

Substitute $a = ub$ yields

$$\sigma_z = \frac{1}{2}\left(\frac{z}{b}\right)^2\left(1-\frac{\rho}{\beta}\right)\frac{q}{SI(\alpha)}\int_{-1}^{1}\frac{\left(\frac{x}{b}-u\right)\sin\alpha u}{\left[\left(\frac{x}{b}-u\right)^2+\left(\frac{z}{b}\right)^2\right]^2}du \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Note: if $b \to \infty$, $\sigma_z = 0$ and if $x = 0$ and $z = 0$ $\sigma_z = 0$ for σ_x yields

$$\sigma_x = \frac{1}{2}\left(1-\frac{\rho}{\beta}\right)\frac{q}{SI(\alpha)}\int_{-b}^{b}\frac{(x-a)\sin\left(\frac{\alpha a}{b}\right)}{(x-a)^2+z^2}da - \sigma_z$$

Substitute $a = ub$ yields

$$\sigma_x = \frac{1}{2}\left(1-\frac{\rho}{\beta}\right)\frac{q}{SI(\alpha)}\int_{-1}^{1}\frac{\left(\frac{x}{b}-u\right)\sin\alpha u}{\left(\frac{x}{b}-u\right)^2+\left(\frac{z}{b}\right)^2}du - \sigma_z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

And as $b \to \infty \sigma_x = -q\left(1-\frac{\rho}{\beta}\right)\frac{SI(\alpha)}{SI(\alpha)} = -q\left(1-\frac{\rho}{\beta}\right)$

.... (32)

And $x = 0$ and $z = 0$ in Eq. 31 $\sigma_x = -q\left(1-\frac{\rho}{\beta}\right)$

Now when including the uniform load q the strain becomes:

$$\sigma_x = q_{\text{VERTICAL LOAD}} - q\left(1-\frac{\rho}{\beta}\right)_{\text{SHEAR LOAD}}$$

$$= \frac{\rho}{\beta}q = K_0 q \quad \text{for} \begin{cases} b \to \infty \\ \text{or} \\ x = z = 0 \end{cases} \text{in Eq. 31}$$

and

$$\varepsilon_x = \left(\beta\sigma_x - \rho\sigma_z\right)_{\text{VERTICAL LOAD}} + \left(\beta\sigma_x - \rho\sigma_z\right)_{\text{SHEAR LOAD}}$$

$$= (\beta-\rho)q - \beta\left(1-\frac{\rho}{\beta}\right) = 0 \quad \text{for} \begin{cases} b \to \infty \\ \text{or} \\ x = z = 0 \end{cases} \text{in Eq. 31}$$

And the criterion discussed in the previous section is satisfied. Since α can take any value the function

$$t(x,\alpha_n) = \frac{\sum \frac{\pi}{4}\left(1-\frac{\rho}{\beta}\right) q \sin\left(\frac{\alpha_n x}{b}\right)}{\sum SI(\alpha_n)}$$

Satisfies the above criterion and it can be concluded that such a shear function not only exists but also a family of them do exist.

<u>2-General Solution:</u> In order to have a true grip of the general solution, the analysis will be based on minimizing the strain energy which can be expressed as:

$$V = \int_{-\infty}^{\infty}\int_{0}^{\infty} V_0 dzdx = \frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \left[\beta\left(\sigma_x^2 + \sigma_z^2\right) - 2\rho\sigma_x\sigma_z + \frac{2\rho}{\nu}\tau_{xz}^2\right] dzdx \quad \ldots\ldots\ldots\ldots\ldots\ldots.. (33)$$

Where V_0 is the strain energy per unit volume, V is the energy per unit length. If considering strictly an even function or an odd function pressure on the semi-infinite solid $q(x)$ the strain energy becomes:

$$V = 2\int_{0}^{\infty}\int_{0}^{\infty} V_0 dzdx = \int_{0}^{\infty}\int_{0}^{\infty} \left[\beta\left(\sigma_x^2 + \sigma_z^2\right) - 2\rho\sigma_x\sigma_z + \frac{2\rho}{\nu}\tau_{xz}^2\right] dzdx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

So by superposition any load can be achieved using an even and an odd function.

This analysis will start first with showing that Eq. 22 is indeed the solution for an infinite uniform load on the surface. Consider the solution for an infinite uniform load as:

$\sigma_z = q, \quad \sigma_x = Kq \quad$ and $\quad \tau_{xz} = 0$

Where K is a constant to be found, if substituting in Eq. 34 yields,

$$V = q^2 \iint \left[\beta\left(K^2 + 1\right) - 2\rho K\right] dz dx \quad \text{.......} \quad (35)$$

Minimizing V with respect to K yields:

$$\frac{\partial V}{\partial K} = 0 \rightarrow 2\beta K - 2\rho = 0 \rightarrow K = \frac{\rho}{\beta} = K_0 \quad \text{.......} \quad (36)$$

Secondly extracting the general solution and showing Kelvin solution is indeed correct.

Consider $q(x)$ an even function as in Fig. 7.

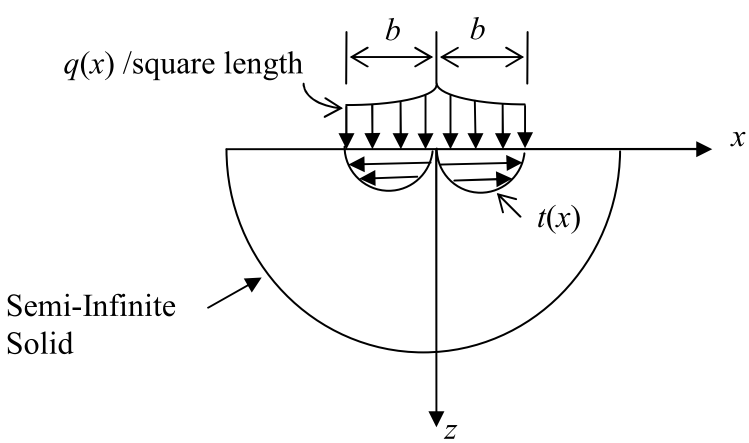

Fig. 6

With the Φ function:

$$\Phi = \int_0^{\infty} \frac{1}{\alpha^2}\left(C + D\alpha z\right)e^{-\alpha z} \cos \alpha x \, d\alpha \quad \text{……………………………………………… (37)}$$

The Bi-harmonic equation and the boundary condition is satisfied:

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial z^2} + \frac{\partial^4 \Phi}{\partial z^4} = 0 \quad \text{………………………………………………………… (38)}$$

The stresses are:

$$\sigma_z = \frac{\partial^2 \Phi}{\partial x^2} = -\int_0^{\infty} \left(C + D\alpha z\right)e^{-\alpha z} \cos \alpha x \, d\alpha \quad \text{………………………………………….. (39)}$$

$$\sigma_x = \frac{\partial^2 \Phi}{\partial z^2} = \int_0^{\infty} \left[\left(C - 2D\right) + D\alpha z\right]e^{-\alpha z} \cos \alpha x \, d\alpha \quad \text{………………………………….. (40)}$$

$$\tau_{xz} = -\frac{\partial^2 \Phi}{\partial z \partial x} = -\int_0^{\infty} \left[\left(C - D\right) + D\alpha z\right]e^{-\alpha z} \sin \alpha x \, d\alpha \quad \text{………………………………….. (41)}$$

Where C and D are constants and can be a function of α to be determined. At $z = 0$ Eq. 39 becomes:

$$\sigma_z\Big|_{z=0} = -\int_0^{\infty} C \cos \alpha x \, dx \quad \text{……………………………………………………….. (42)}$$

By taking the Fourier transform of $q(x)$, assuming $q(x)$ is an even function and satisfies the Fourier Integral criterion, yield:

$$q(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x \, d\alpha \int_0^\infty q(\lambda)\cos\alpha\lambda \, d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(43)}$$

Equating Eq. 42 to 43, yields:

$$C(\alpha) = -\frac{2}{\pi}\int_0^b q(\lambda)\cos\alpha\lambda \, d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(44)}$$

Now substituting Eq. 39 through Eq. 41 in Eq. 34 and investigating the first term of the energy function yields:

$$\begin{aligned}
\int_0^\infty dx \int_0^\infty dz \, \sigma_z^2 &= \int_0^\infty dx \int_0^\infty dz \left[\int_0^\infty (C + D\alpha z)e^{-\alpha z} \cos\alpha x \, d\alpha\right]^2 \\
&= \int_0^\infty dx \int_0^\infty dz \left[\int_0^\infty (C + D\xi z)e^{-\xi z} \cos\xi x \, d\xi\right]\left[\int_0^\infty (C + D\alpha z)e^{-\alpha z} \cos\alpha x \, d\alpha\right]
\end{aligned}$$

$$= \int_0^\infty dx \int_0^\infty d\alpha \int_0^\infty d\xi \left[\frac{C(\alpha)C(\xi)}{\alpha+\xi} + \frac{C(\alpha)D(\xi)}{(\alpha+\xi)^2}\xi + \frac{C(\xi)D(\alpha)}{(\alpha+\xi)^2}\alpha + 2\frac{D(\alpha)D(\xi)}{(\alpha+\xi)^3}\alpha\xi\right]\cos\xi x \, \cos\alpha x$$

Noting the first term of the integration represent a Fourier series as

$$\lim_{\varphi \to \alpha} \frac{\pi}{2}\int_0^\infty C(\alpha)\left[\frac{2}{\pi}\int_0^\infty \cos\alpha x \, dx \int_0^\infty \frac{C(\xi)}{\varphi+\xi}\cos\xi x \, d\xi\right] d\alpha = \frac{\pi}{2}\int_0^\infty \frac{[C(\alpha)]^2}{\varphi+\alpha} d\alpha = \frac{\pi}{4}\int_0^\infty \frac{[C(\alpha)]^2}{\alpha} d\alpha$$

This is ok as long as

$\left| \int_0^\infty \frac{C(\xi)}{\varphi + \xi} \right| < \infty$ for the proper choice of *C*. It can be easily be shown for $q(x) = q$ constant

this can be satisfied. From this relation yields:

$$\int_0^\infty dx \int_0^\infty dz\, \sigma_z^2 = \frac{\pi}{2} \int_0^\infty \left[\frac{C^2}{2\alpha} + \frac{CD}{2\alpha} + \frac{D^2}{4\alpha} \right] d\alpha = \frac{\pi}{4} \int_0^\infty \frac{1}{\alpha} \left[C^2 + CD + \frac{D^2}{2} \right] d\alpha$$

Repeating this process to the rest of the terms of the energy equation, Eq. 34, yields:

$$V = \frac{\pi}{4} \int_0^\infty \left[\beta\left(2C^2 - 2CD + 3D^2\right) + \rho\left(2C^2 - 2CD - D^2\right) + \frac{\rho}{\nu}\left(2C^2 - 2CD + D^2\right) \right] \frac{d\alpha}{\alpha} \quad \text{……………………………. (45)}$$

Now let $D - C = D_0$ and substitute then

$$V = \frac{\pi}{4} \int \frac{1}{\alpha} \left[-2\left(\beta + \rho + \frac{\rho}{\nu} \right) CD_0 + \left(3\beta - \rho + \frac{\rho}{\nu} \right) (C + D_0)^2 \right] d\alpha$$

For which it can be re-written as:

$$V = \pi(1+\nu) \int_0^\infty \frac{1}{\alpha} \left[-CD_0 + (1-\nu)(C + D_0)^2 \right] d\alpha \quad \text{…………………………..………… (47)}$$

V can be minimized two ways: 1- with the proper choice of D_0 2- with the proper choice of $t(x)$.

First case minimize with respect to D_0:

$$\frac{\partial V}{\partial D_0} = 0 \Rightarrow -C + 2(1-\nu)(C + D_0) = 0 \Rightarrow D_0 = \frac{-1+2\nu}{2(1-\nu)}C$$

So that

$$D = C + D_0 = \frac{C}{2(1-\nu)} \quad \ldots\ldots\ldots\ldots (48)$$

The first consequence to this choice of D is the strain ε_x at the surface is zero everywhere. Since

$$\varepsilon_x = \beta\sigma_x - \rho\sigma_z = \beta\int_0^\infty \left[(C-2D) + D\alpha z\right]e^{-\alpha z}\cos\alpha x \, d\alpha - \rho\int_0^\infty -(C + D\alpha z)e^{-\alpha z}\cos\alpha x \, d\alpha$$

At $z = 0$ the equation becomes:

$$\varepsilon_x = \int_0^\infty \left[C(\beta+\rho) - 2\beta D)\right]\cos\alpha x \, d\alpha = 0 \Rightarrow D = \frac{\beta+\rho}{2\beta}C = \frac{C}{2(1-\nu)}$$

Which corresponds to Eq. 48. The second consequence to this choice of D_0 is if one chooses $q(\lambda) = q$ constant then from Eq. 44 yields:

$$C(\alpha) = -\frac{2}{\pi}\frac{\sin \alpha b}{\alpha} q \quad \text{.......} \quad (49)$$

$$D(\alpha) = -\frac{2}{\pi}\left[\frac{1}{2(1-\nu)}\right]\frac{\sin \alpha b}{\alpha} q \quad \text{.......} \quad (50)$$

When substituting Eq. 49 and Eq. 50 in Eq. 39, 40 and Eq. 41 and integrating the Laplace transform the result gives the same equation as Eq. 5, 6 and Eq.8. Also, it can be shown that the solution for a point load of Eq. 1, 2 and Eq. 4 can be obtained by letting b go to zero with qb remain constant. Furthermore the derivation shown above was not derived by the use of Boussineque equations in which Kelvin solution was originally derived from. The derivation was derived from minimizing the strain energy and shows that ε_x at the surface is absolutely equal zero.

<u>Second case minimize with respect to *t*(*x*):</u>

With $C(\alpha) = -\frac{2}{\pi}\int_0^b q(\lambda)\cos \alpha\lambda \, d\lambda$, choosing

$$D_0(\alpha) = \frac{2}{\pi}\int_0^b t(\lambda)\sin \alpha\lambda \, d\lambda = D - C \quad \text{.......} \quad (51)$$

So that at $z = 0$ in Eq. 41, $\tau_{xz} = \int_0^\infty D_0 \sin \alpha x \, d\alpha$

Substituting C and this choice of D_0 in Eq. 47, yields:

$$V = \frac{1+\nu}{\pi}\int_0^\infty \left\{ \frac{1}{\alpha}\left[\int_0^b q(\lambda)\cos\alpha\lambda\, d\lambda\right]\left[\int_0^b t(\lambda)\sin\alpha\lambda\, d\lambda\right] + \frac{1-\nu}{\alpha}\left[\int_0^b t(\lambda)\sin\alpha\lambda\, d\lambda - \int_0^b q(\lambda)\cos\alpha\lambda\, d\lambda\right]^2 \right\} d\alpha$$

Now

$$\frac{\partial V}{\partial t} = 0 = \frac{1+\nu}{\pi}\int_0^\infty \left\{ \frac{2\nu-1}{\alpha}\left[\int_0^b q(\lambda)\cos\alpha\lambda\, d\lambda\right]\left[\int_0^b \sin\alpha\lambda\, d\lambda\right] + 2\frac{1-\nu}{\alpha}\left[\int_0^b t(\lambda)\sin\alpha\lambda\, d\lambda\right]\left[\int_0^b \sin\alpha\lambda\, d\lambda\right] \right\} d\alpha$$

$$0 = \frac{2\nu-1}{2}\pi\int_0^b \left\{ \int_0^\xi \left[\frac{2}{\pi}\int_0^\infty \cos\alpha r\, d\alpha \int_0^b q(\lambda)\cos\alpha\lambda\, d\lambda\right] dr \right\} d\xi + 2(1-\nu)\int_0^b t(\lambda)\, d\lambda \int_0^b d\xi \int_0^\infty \frac{\sin\alpha\lambda\sin\alpha\xi}{\alpha}\, d\lambda$$

$$0 = \frac{2\nu-1}{2}\pi\int_0^b \left[\int_0^\xi q(r)\, dr\right] d\xi + (1-\nu)\int_0^b t(\lambda)\, d\lambda \int_0^b \ln\left|\frac{\lambda+\xi}{\lambda-\xi}\right| d\xi$$

Or

$$0 = \int_0^b \left\{ \frac{1-2\nu}{2(1-\nu)}\pi\int_0^\xi q(r)\, dr - \int_0^b t(\alpha)\ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right| d\alpha \right\} d\xi \quad \text{.................................} \quad (51A)$$

Thus

$$\frac{\pi}{2}\left(\frac{\beta\text{ - }\rho}{\beta}\right)\int_0^{\xi} q(r)\,dr = \int_0^b t(\alpha)\ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right|d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(51B)}$$

Which can be written as:

$$\frac{\pi}{2}\left(\frac{\beta\text{ - }\rho}{\beta}\right)\int_0^{\psi} q(bu)\,du = \int_{-1}^{1} t(b\lambda)\ln|\lambda - x|d\lambda \qquad \text{where} \quad \psi = \frac{\xi}{b}$$

This integral equation is solved by Wiener-Hopf procedure [4] with second term is eliminated for an even function *q*(*x)* and the result is

$$t(b\psi) = -\frac{(\beta-\rho)}{2\pi\beta\sqrt{1-\psi^2}}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}}{\xi-\psi}q(b\xi)d\xi + t_0$$

Which leads to

$$t(x) = -\frac{(\beta-\rho)}{2\pi\beta\sqrt{b^2-x^2}}\int_{-b}^{b}\frac{\sqrt{b^2-\lambda^2}}{\lambda-x}q(\lambda)d\lambda + t_0(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(52)}$$

Where $t_0(x)$ must satisfy the homogeneous solution condition in Eq. 51A

$$\int_0^b t_0(\alpha)\int_0^b \ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right|d\xi d\alpha \equiv 0$$

$t(x)$ in Eq. 52 shall be defined as the contact shear. $t_0(x)$ is a function that need to satisfy compatibility between both materials, the semi-infinite solid and the material transmitting the load. $t(x)$ may not minimize the strain energy in the material transmitting the load. However, $t_0(x)$ may help do that.

Note at $x = b$ in Eq. 52 the contact shear is infinite which will give $\sigma_x \to \infty$ at $x = b^+$. This is common when introducing any shear load at the surface to have infinite lateral stress at the corners of the load at the surface. In this case plasticity occurs. However, the contact shear it self being infinite may not possible physically, because may have cracks at the corner of the load every time the load is applied with hard material. If restrict the shear at some distance b_0 then Eq. 52 can be applied at $b_0 \le x \le b_0$ where $b_0 \to b$ or b_0 approach b. Then Eq. 51 becomes:

$$D_0(\alpha) = \frac{2}{\pi}\int_0^{b_0} t(\lambda)\sin\alpha\lambda\, d\lambda = D - C$$

And Eq. 51B becomes:

$$\frac{\pi}{2}\left(\frac{\beta - \rho}{\beta}\right)\int_0^{\xi} q(r)\,dr = \int_0^{b_0} t(\alpha)\ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right| d\alpha = \int_0^{b} t(\alpha)\ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right| d\alpha - \int_{b_0}^{b} t(\alpha)\ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right| d\alpha$$

or

$$\int_{b_0}^{b} t(\alpha)\ln\left|\frac{\xi+\alpha}{\xi-\alpha}\right| d\alpha \to 0 \quad \text{as} \quad b_0 \to b$$

Thus the strain energy is minimized to the best it can from a physical sense.

Consequence that effects Melan's Solution:

The first consequence is that the method of subtraction Melan outlined must be updated and his solution must be corrected when *b* approach infinity. This is becomes necessary because when using Bousineesque equations for a point load (not a shear point load) to subtract the stress from the surface he did not take into account of the contact shear where it makes zero strain at the center of the load and the deflection in the *x* direction at the surface zero *b* approach infinity. Thus, the deflection derivation is as follows:

$\varepsilon_z = \beta\sigma_z - \rho\sigma_x$ and $\varepsilon_x = \beta\sigma_x - \rho\sigma_z$ so that :

$$\varepsilon_z = \frac{\partial v}{\partial z} = \int_0^\infty \{(\beta+\rho)C - 2\rho D + (\beta+\rho)D\alpha z\} e^{-\alpha z} \cos\alpha x d\alpha$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \int_0^\infty \{(\beta+\rho)C - 2\beta D + (\beta+\rho)D\alpha z\} e^{-\alpha z} \cos\alpha x d\alpha$$

$$v = \int_0^\infty \left\{ \frac{(\beta+\rho)C - 2\rho D}{-\alpha} + \frac{(\beta+\rho)D(\alpha z+1)}{-\alpha} \right\} e^{-\alpha z} \cos\alpha x d\alpha + g(x)\Big|_{\to 0\, for\, @\, z\to\infty\, v=0}$$

$$u = \int_0^\infty \{(\beta+\rho)C - 2\beta D + (\beta+\rho)D\alpha z\} e^{-\alpha z} \frac{\sin\alpha x}{\alpha} d\alpha + f(z)$$

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}=-2(\beta+\rho)\int_0^{\infty}\{C-D+D\alpha z\}e^{-\alpha z}\sin\alpha x\,d\alpha+\dot{f}(z)=\frac{\tau_{xz}}{G}=\frac{2(1+\nu)}{E}\tau_{xz}=2(\beta+\rho)\tau_{xz}$$

From Eq. 41 and at $x = 0$ $u = 0$, $f(z) = 0$ thus

$$v=\int_0^{\infty}\{(\beta+\rho)C+(\beta-\rho)D+(\beta+\rho)D\alpha z\}e^{-\alpha z}\frac{\cos\alpha x}{\alpha}d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

$$u=\int_0^{\infty}\{(\beta+\rho)C-2\beta D+(\beta+\rho)D\alpha z\}e^{-\alpha z}\frac{\sin\alpha x}{\alpha}d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

At $z = 0$ yields:

$$u\big|_{z=0}=\int_0^{\infty}\{(\beta+\rho)C-2\beta D\}\frac{\sin\alpha x}{\alpha}d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

Now substitute D = C + D_0 in Eq. 55 yields

$$u\big|_{z=0}=\int_0^{\infty}\{(\rho-\beta)C-2\beta D_0\}\frac{\sin\alpha x}{\alpha}d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

Then substitute Eq. 44 and Eq. 56 with changing to the full integrand yields:

$$u\big|_{z=0}=-\frac{1}{\pi}\int_0^{\infty}\left\{(\rho-\beta)\int_{-b}^{b}q(\lambda)\cos\alpha\beta\lambda\,d\lambda-2\beta\int_{-b_0}^{b_0}t(\lambda)\sin\alpha\beta\lambda\,d\lambda\right\}\frac{\sin\alpha x}{\alpha}d\alpha$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57)$$

Using the Fourier Transform on the left side and integrating the right side yields:

$$u\big|_{z=0} = -\frac{1}{2}(\rho-\beta)\begin{Bmatrix} \int_{-x}^{x} q(\lambda)d\lambda \quad \text{for } x^2 \le b^2 \\ \text{zero} \quad \text{for } x^2 > b^2 \end{Bmatrix} - \frac{\beta}{\pi}\int_{-b_0}^{b_0} t(\lambda)\ln\left|\frac{x+\lambda}{x-\lambda}\right| \quad \text{.....................} (58)$$

Let $u = 0$ in the region $-b \le x \le b$ with $b_0 = b$ Eq. 51B can be obtained

And the minimum energy criterion is satisfied and the choice in Eq. 52 make the lateral deflection underneath the load close to zero. If substitute for a constant load q in Eq. 58 yields:

$$u\big|_{z=0} = (\beta-\rho)\begin{Bmatrix} -q|x| \quad \text{for } x^2 \le b^2 \\ \text{zero} \quad \text{for } x^2 > b^2 \end{Bmatrix} - (\beta-\rho)\begin{Bmatrix} -q|x| \quad \text{for } x^2 \le b^2 \\ -q\left(|x| - \sqrt{x^2-b^2}\right) \quad \text{for } x^2 > b^2 \end{Bmatrix}$$

or

$$u\big|_{z=0} = (\beta-\rho)\begin{Bmatrix} \text{zero} \quad \text{for } x^2 \le b^2 \\ q\left(|x| - \sqrt{x^2-b^2}\right) \quad \text{for } x^2 > b^2 \end{Bmatrix}$$

.... (59)

Where the sign change to account for compression and the contact shear is found from Eq. 52 yield:

$$t(x) = -\frac{(\beta-\rho)q}{2\pi\beta\sqrt{b^2-x^2}}\int_{-b}^{b}\frac{\sqrt{b^2-\lambda^2}}{\lambda-x}\,d\lambda = \frac{(\beta-\rho)q}{2\beta}\frac{x}{\sqrt{b^2-x^2}}$$

……………………………………….. (59A)

For the strain from Eq. 57 and Eq. 58 yields at $z = x = 0$:

$$\begin{aligned}
\varepsilon\big|_{z=x=0} = \frac{\partial u}{\partial x}\bigg|_{z=x=0} &= -\frac{1}{\pi}\int_0^{\infty}\left\{(\rho-\beta)\int_{-b}^{b} q(\lambda)\cos\alpha\beta\lambda\,d\lambda - 2\beta\int_{-b_0}^{b_0} t(\lambda)\sin\alpha\beta\lambda\,d\lambda\right\}d\alpha \\
&= (\beta-\rho)q(0) - \frac{2\beta}{\pi}\int_{-b_0}^{b_0}\frac{t(\lambda)}{\lambda}\,d\lambda \\
&= (\beta-\rho)q - \frac{(\beta-\rho)q}{\pi}\int_{-b_0}^{b_0}\frac{1}{\sqrt{b^2-\lambda^2}}\,d\lambda \quad \text{for} \quad q(x) = q \\
&= (\beta-\rho)q - \frac{2(\beta-\rho)q}{\pi}\sin^{-1}\left(\frac{b_0}{b}\right) \\
&= 0 \quad \text{as} \quad b_0 \to b
\end{aligned}$$

Thus the strain at $z = x = 0$ is close to zero and not $q(\beta-\rho)$

Note that if the load has slippage at the outside as in a pocking surface and the deflection at $-\delta \le x \le \delta$, where δ is small, is close to zero, the strain $\varepsilon_x = 0$ for a smaller $b = \delta$. It is expected two small cracks or plasticity at $x = b$ and $x = -b$ of the load due to infinite lateral stress as seen when walking on soft moist soil.

By inspection the vertical deflection under the load is not constant since Eq. 53 is a function. $\sigma_z = q$ and $\sigma_x = K_0 q$ are constant at $-b \le x \le b$ and a rigid body deformation occur that is able to flex under the load for the selected contact shear.

Solution for Uniform load

Adding the contact shear using Boussinesque equation for shear yields:

$$\sigma_z = \frac{2}{\pi} z^2 \int_{-b}^{b} t(a) \frac{(x-a)}{\left[(x-a)^2 + z^2\right]^2} da \quad \text{.....} \quad (60)$$

$$\sigma_x = \frac{2}{\pi} \int_{-b}^{b} t(a) \frac{(x-a)^3}{\left[(x-a)^2 + z^2\right]^2} da \quad \text{.....} \quad (61)$$

$$\tau_{xz} = \frac{2}{\pi} z \int_{-b}^{b} t(a) \frac{(x-a)^2}{\left[(x-a)^2 + z^2\right]^2} da \quad \text{.....} \quad (62)$$

Substitute Eq. 59A and let $M = \frac{x}{b}$, $N = \frac{z}{b}$, $a = b\sin\theta$ and $da = b\cos\theta d\theta$ in Eq. 60, 61 and 62 yields:

$$\sigma_z = \frac{\beta - \rho}{\pi\beta} q N^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta(M - \sin\theta)}{\left[(M - \sin\theta)^2 + N^2\right]^2} d\theta \quad \text{.....} \quad (63)$$

$$\sigma_x = \frac{\beta - \rho}{\pi\beta} q \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta(M - \sin\theta)^3}{\left[(M - \sin\theta)^2 + N^2\right]^2} d\theta \quad \text{.....} \quad (64)$$

$$\tau_{xz} = \frac{\beta - \rho}{\pi\beta} qN \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta (M - \sin\theta)^2}{\left[(M - \sin\theta)^2 + N^2\right]^2} d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(65)}$$

Integrating Eq. 63 through 65 using complex number in the denominators yields:

$$\sigma_z = \frac{\beta - \rho}{2\beta} \frac{Nq}{\left(M^2 + N^2\right)^{\frac{1}{2}} \left[\left(M^2 - N^2 - 1\right)^2 + 4M^2N^2\right]^{\frac{5}{4}}}$$

$$\times \left[M\left(3N^2 - M^2 + 1\right)\sin\left(\theta - \frac{\psi + \varphi}{2}\right) - N\left(N^2 - 3M^2 + 1\right)\cos\left(\theta - \frac{\psi + \varphi}{2}\right)\right] \quad \ldots \text{(66)}$$

$$\sigma_x = \frac{\beta - \rho}{\beta} q\left\{ -1 + \frac{\left(M^2 + N^2\right)^{\frac{1}{2}}}{\left[\left(M^2 - N^2 - 1\right)^2 + 4M^2N^2\right]^{\frac{1}{4}}} \cos\left(\theta - \frac{\psi + \varphi}{2}\right)\right\} - \sigma_z \quad \ldots\ldots\ldots\ldots \text{(67)}$$

$$\tau_{xz} = \left[-\frac{\beta-\rho}{\beta} q \frac{\left(M^2+N^2\right)^{\frac{1}{2}}}{\left[\left(M^2-N^2-1\right)^2+4M^2N^2\right]^{\frac{1}{4}}} + \frac{\beta-\rho}{2\beta} q \frac{\left(M^2+N^2\right)^3-2M^4+3M^2N^2+N^4+M^2}{\left(M^2+N^2\right)^{\frac{1}{2}}\left[\left(M^2-N^2-1\right)^2+4M^2N^2\right]^{\frac{5}{4}}} \right]$$

$$\times \sin\left(\theta-\frac{\psi+\varphi}{2}\right) - \frac{\beta-\rho}{2\beta} q \frac{N\left(M^2+N^2\right)^{\frac{1}{2}}}{\left[\left(M^2-N^2-1\right)^2+4M^2N^2\right]^{\frac{1}{4}}} \left[\frac{M}{M^2+N^2} - \frac{M\left(M^2+N^2-1\right)}{\left(M^2-N^2-1\right)^2+4M^2N^2} \right]$$

$$\times \cos\left(\theta-\frac{\psi+\varphi}{2}\right)$$

……………………………………………….. (68)

$$\sigma_y = \sigma_z = \frac{\beta-\rho}{2\beta} \frac{Nq}{\left(M^2+N^2\right)^{\frac{1}{2}}\left[\left(M^2-N^2-1\right)^2+4M^2N^2\right]^{\frac{5}{4}}}$$

…… (69)

$$\times \left[M\left(3N^2-M^2+1\right)\sin\left(\theta-\frac{\psi+\varphi}{2}\right) - N\left(N^2-3M^2+1\right)\cos\left(\theta-\frac{\psi+\varphi}{2}\right) \right]$$

Equations 66, 67, 68 and 69 must be added to Eq. 15, 16, 17 and 18 respectively.

Where θ, ψ and φ are the angles in Fig 7. Fig. 8 and table 1 shows the lateral stress σ_x for $\nu = 0.35$ and $q = 1.0$. Note σ_x under the load at $z = 0$ is not q as in Bousinesseque equation. The difference can be as much as 50% depending on poison's ratio. This problem can also happen in finite element when ignoring the contact shear. It maybe wise to use the maximum shear derived in Eq. 52 and compare with no contact shear.

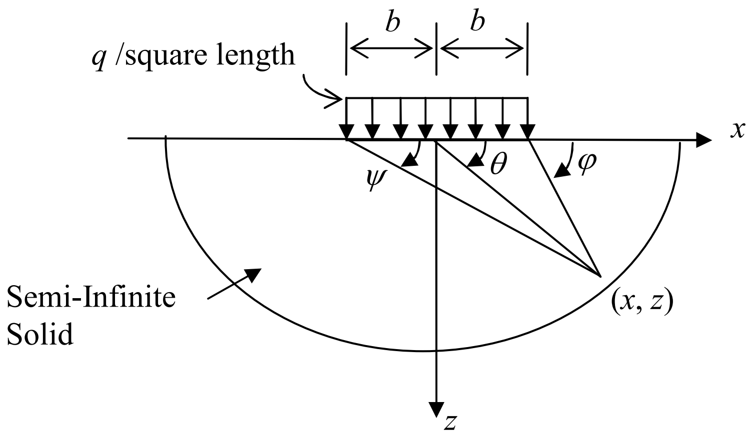

Fig. 7

Table 1

N	M = 0	M = 0.5	M = 1	M = 1.5	M = 4	M = 10
0.0	0.53846	0.53846	-----------	0.15768	0.01514	0.00233
0.1	0.48064	0.47566	0.62432	0.24495	0.02355	0.00361
0.2	0.42440	0.41522	0.46408	0.29802	0.03176	0.00489
0.3	0.37116	0.35966	0.38766	0.32028	0.03971	0.00616
0.4	0.32199	0.31075	0.33756	0.32181	0.04733	0.00743
0.5	0.27762	0.26889	0.29976	0.31183	0.05456	0.00869
0.6	0.23832	0.23348	0.26902	0.29635	0.06137	0.00993
0.7	0.20407	0.20357	0.24294	0.27872	0.06771	0.01117
0.8	0.17459	0.17820	0.22022	0.26066	0.07355	0.01239
0.9	0.14943	0.15655	0.20012	0.24301	0.07888	0.01360
1.0	0.12810	0.13797	0.18218	0.22618	0.08367	0.01479
1.1	0.11008	0.12195	0.16605	0.21032	0.08794	0.01596
1.2	0.09487	0.10807	0.15151	0.19549	0.09167	0.01712
1.3	0.08205	0.09601	0.13837	0.18167	0.09489	0.01825
1.4	0.07123	0.08552	0.12648	0.16883	0.09760	0.01936
1.5	0.06207	0.07635	0.11572	0.15693	0.09983	0.02046
1.6	0.05431	0.06833	0.10597	0.14590	0.10160	0.02152
1.7	0.04770	0.06129	0.09713	0.13569	0.10293	0.02257
1.8	0.04206	0.05511	0.08912	0.12625	0.10386	0.02359

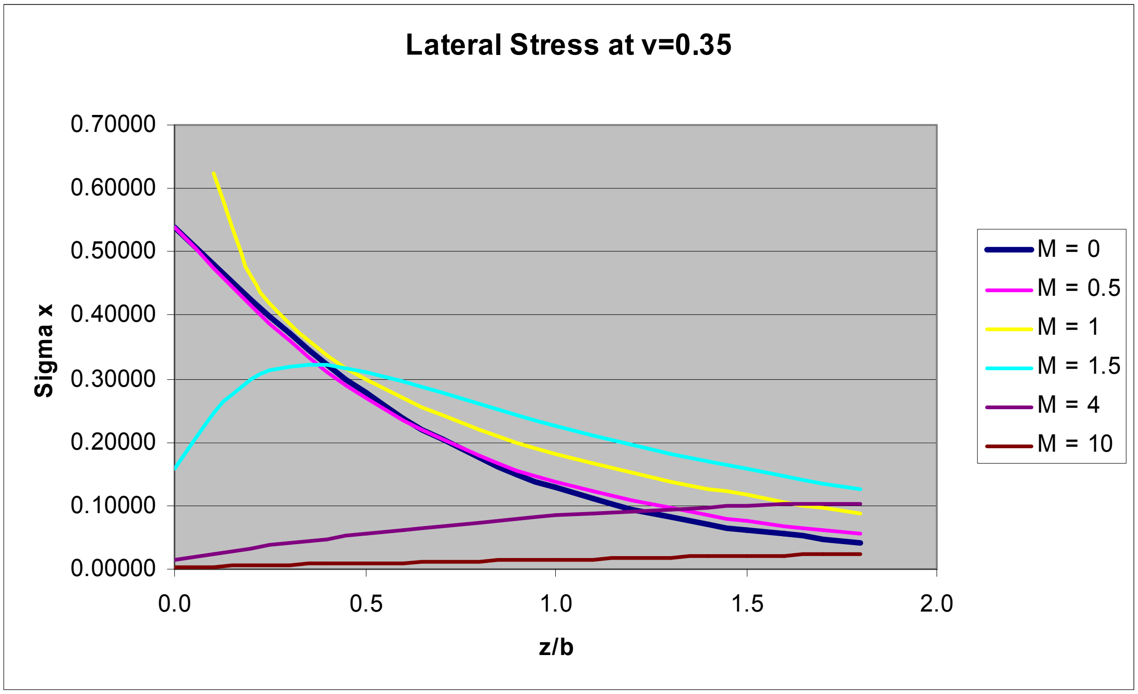

Fig 8

Contact Shear for a slab with a load:

If a slab deflecting *PL*/*AE* then the deflection must match between the slab and the semi-infinite solid. Rewriting Eq. 58 yields:

$$u\big|_{z=0} = \frac{1}{2}(\beta-\rho)\int_{-x}^{x} q(\lambda)d\lambda - \frac{\beta}{\pi}\int_{-b}^{b} t(\lambda)\ln\left|\frac{x+\lambda}{x-\lambda}\right| = \frac{x\int_0^x t(\lambda)d\lambda}{AE_0} \quad\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

Where E_0 is the elastic modulus of the slab and *A* is the cross section area. Using series solution Eq. 70 can be solved let:

$$q(\lambda) = \sum_{n=0}^{\infty} c_n \left(\frac{\lambda}{b}\right)^{2n}$$

or …………………………………………………………. (71)

$$\int_{-x}^{x} q(\lambda) d\lambda = 2b \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \left(\frac{x}{b}\right)^{2n+1}$$

And

$$t(\lambda) = \sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{b}\right)^{2n+1}$$

or …………………………….. (72)

$$\int_{-b}^{b} t(\lambda) \ln\left|\frac{x+\lambda}{x-\lambda}\right| = b \sum_{n-0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m}{(2n+1)(2m-2n+1)} \left(\frac{x}{b}\right)^{2n+1}$$

And from Eq. 72

$$x \int_{0}^{x} t(\lambda) d\lambda = b^2 \sum_{n=0}^{\infty} a_n \left(\frac{x}{b}\right)^{2n+3} \quad \text{…………………………………………………………. (73)}$$

Substitute Eq. 71, 72 and 73 in Eq. 70 yields:

$$(\beta - \rho) \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \left(\frac{x}{b}\right)^{2n+1} - \frac{\beta}{\pi} \sum_{n-0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m}{(2n+1)(2m-2n+1)} \left(\frac{x}{b}\right)^{2n+1} = \frac{b \sum_{n=0}^{\infty} a_n \left(\frac{x}{b}\right)^{2n+3}}{AE_0} \quad \text{.. (74)}$$

Eq. 74 creates a matrix inversion solution for finding the coefficients a_n as a function on c_n when equating term by term and it is solvable. It is expected when b goes to infinity the contact shear goes to zero and the strain under the surface goes to zero.

Applications 1:

Vehicle tires rotate on the road because of contact shear and normal times coefficient of friction. For example if the road is also made out of rubber and poison's ratio is 0.5 it would not get no contact shear just normal times coefficient of friction. If assume the strain is zero under the tire then an approximate locale plain strain problem occur. For the purpose of estimating the magnitude of the contact shear assume a uniform strip load as follows.

Thus

$$t(x) = \frac{(\beta - \rho)qx}{2\beta\sqrt{b^2 - x^2}} = (1 - K_0)\frac{q}{2}\frac{x}{\sqrt{b^2 - x^2}} = (1 - K_0)\frac{P}{4b_0}\frac{x}{\sqrt{b^2 - x^2}} \quad \ldots\ldots\ldots\ldots\ldots.. (75)$$

Where P is the axial load per ft and b_0 is one half the actual contact distance of the load. If assume maximum shear allowed at b_0 yields

$$\tau_{\max} = (1-K_0)\frac{P}{4b_0}\frac{b_0}{\sqrt{b^2-b_0^2}}$$

or .... (76)

$$b = \sqrt{\left[\frac{P(1-K_0)}{4\tau_{\max}}\right]^2 + b_0^2}$$

The total shear becomes

$$S = \int_0^{b_0} t(x)dx = (1-K_0)\frac{P}{4b_0}\left[b - \sqrt{b^2-b_0^2}\right] \text{ .... (77)}$$

Substituting Eq. 76 in Eq. 77 yields,

$$S = (1-K_0)\frac{P}{4}\left[\sqrt{\left(\frac{P(1-K_0)}{4b_0\tau_{\max}}\right)^2 + 1} - \frac{P(1-K_0)}{4b_0\tau_{\max}}\right]$$

Note as b_0 increase S increase and there is more traction as the contact area increase a more grip to the road in static condition.

<u>Applications 2:</u>

The Hertz contact stress between two rolling cylinder to cylinder surface is expected to have additional shear stress due to contact shear and σ_x at the surface is less than σ_z and not equal. The Hertz net pressure in the *x* direction

$$q(x) = q\sqrt{1 - \frac{x^2}{b^2}}$$

The maximum contact shear stress from Eq. 52 is

$$t(x) = -\frac{(\beta - \rho)}{2\pi\beta\sqrt{b^2 - x^2}} \int_{-b}^{b} \frac{\sqrt{b^2 - \lambda^2}}{\lambda - x} q(\lambda) d\lambda = -\frac{(\beta - \rho)q}{2\pi\beta b\sqrt{b^2 - x^2}} \int_{-b}^{b} \frac{b^2 - \lambda^2}{\lambda - x} d\lambda$$

$$t(x) = \frac{(\beta - \rho)q}{2\pi\beta b\sqrt{b^2 - x^2}} \left[2bx - (b^2 - x^2)\ln\left|\frac{b - x}{b + x}\right| \right]$$

Conclusion:

The contact shear has been derived for a strip load and integrating Boussinesque equation to a uniform load is addressed. Because the contact shear introduce infinite stress if rigid body deterioration of materials can occur as seen in gears subject to constant static contact. If shear traction is to be measured in the laboratory the simple formula is suggested

$$S = A(1 - K_0)q = A'(1 - K_0)\left(\frac{P}{2b_0}\right)$$

Where *A* and *A'* is a coefficient to be measured in the laboratory, q is the average stress on the contact surface. P is the total point line load and b_0 is the half distance of the contact surface. It can be concluded that the coefficient of friction will drop once the body moves into dynamics due to loss of contact shear from the grip of the two materials.

Reference:

1- Lord Kelvin (née William Thomson, physicist, 1824-1907) Proc. Roy. Soc. 14,. 289. (1865).

2- Boussinesq, J., *Application des Potentiels à L'Édute de L'Équilibre et due Mouvement des Solides Élastiques,* Gauthier-Villars, Paris. 1885

3- Melan, E., Z. Angew. Math. Mech., 12, 343 (1932).

4- Carl E. Pearson, Handbook of Applied Math, 2nd ed.,Van Norstrand Reinhold, New York, 1983